

Nontriviality of the linear sigma model

L.R. Babukhadia, M.D. Scadron

Physics Department, University of Arizona, Tucson, AZ 85721, USA

Received: 9 November 1998 / Published online: 11 March 1999

Abstract. We consider techniques (based on an ultraviolet cutoff) used to prove that the pure boson $(\phi^4)_4$ field theory is trivial and apply them instead to the dynamically generated quark-level linear sigma model. This cutoff approach leads to the conclusion that the latter field theory is in fact nontrivial.

1 Introduction

Owing to the recent observational identification [1] of a nonstrange scalar σ meson below 1 GeV, formal field theories discarding such a scalar σ due to "triviality" theorems (meaning the meson-meson coupling $\lambda \rightarrow 0$ when cutoff $\Lambda \rightarrow \infty$) should be reanalyzed as well. In the present paper we show that the quark-level Linear Sigma Model ($L\sigma M$) is a non-trivial field theory in *contrast* with the possibly trivial pure boson $(\lambda\phi^4)_4$ theory. Prior studies of $\lambda\phi^4$ field theory using perturbative and partially nonperturbative methods [2-4] extracted physical constraints on a scalar meson mass via renormalization group bounds and scaling laws [2]. In a somewhat different manner, there are studies of the triviality problem of $(\lambda\phi^4)_4$ theory exploiting a new non-perturbative expansion of the n-point Green's functions [5,6].

Alternatively, one can look at $\lambda\phi^4$ theory also including fundamental fermions — the $L\sigma M$. The key to understanding nontriviality of the quark-level $L\sigma M$ is the Goldberger-Treiman Relation (GTR) which must hold at quark level $m_q = f_\pi g$ to ensure conservation of the axial vector current. It turns out that the bosonic sector of the $L\sigma M$ theory is totally driven by the dynamically generated quark mass via the quark-level GTR demanding even fixed and also non-trivial numerical values for the chiral couplings *independent of any UV cutoff* [7]. Thus the presence of the fundamental $L\sigma M$ fermion fields eliminates the possibility for the chiral couplings to vanish and induces a non-trivial field theory.

In Sect. 2 we summarize recent results for the dynamically generated quark-level $L\sigma M$. Then in Sect. 3 we review both perturbative and nonperturbative techniques for solving the problem of triviality for the pure boson $(\lambda\phi^4)_4$ theory. Finally, in Sect. 4 we demonstrate non-triviality of the quark-level $L\sigma M$. Our results are summarized in Sect. 5. In the appendix we review regularization schemes for the quark-level $L\sigma M$.

2 Dynamically generated quark-level $L\sigma M$

It has been shown recently [7] that the interacting part of the *dynamically generated* $SU(2)$ quark-level $L\sigma M$ lagrangian shifted around the true vacuum, with expectation values $\langle \pi \rangle = \langle \sigma \rangle = 0$ is given by:

$$\mathcal{L}_{L\sigma M}^{Int.} = g' \sigma (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2 + g \bar{\psi} (\sigma + i\gamma_5 \tau \cdot \pi) \psi, \quad (1)$$

with the Gell-Mann-Lévy chiral couplings [8]:

$$g = \frac{m_q}{f_\pi} \quad \text{and} \quad g' = \frac{m_\sigma^2}{2f_\pi} = \lambda f_\pi, \quad (2)$$

and with $m_\pi = 0$. Here, the chiral-limiting pion decay constant $f_\pi \approx 90$ MeV is generated through a *logarithmically* divergent quark loop (Fig. 1). Using Feynman rules for Fig. 1 and the quark-level Goldberger-Treiman Relation (GTR) $f_\pi g = m_q$, one is led to the following logarithmically divergent gap equation

$$1 = -4iN_c g^2 \int \frac{d^4 p}{(p^2 - m_q^2)^2}, \quad (3)$$

where N_c is color number and $d^4 p \equiv d^4 p / (2\pi)^4$.

A quark mass m_q , however, is generated by the *quadratically* divergent tadpole diagram of Fig. 2. Once the $L\sigma M$ is dynamically induced by Figs. 1 and 2, such divergent graphs must be supplemented by σ (shifted field) and π mediating quark self-energies which sum to zero [7]. Moreover, the resulting $L\sigma M$ one-loop order bubble plus tadpole graphs representing m_π^2 sum to zero (as they must by the Goldstone theorem). In addition, the Lee null tadpole condition [9], summing the quark plus σ plus π tadpole graphs to zero should also hold in the $L\sigma M$.

Taking into account dynamically generated meson interactions, one should verify that Lee's null tadpole condition holds for the shifted field σ . This means that the sum of the tadpole graphs of Fig. 3 must vanish:

$$\langle \sigma \rangle = 0 = -8iN_c g m_q \int \frac{d^2 l}{p^2 - m_q^2} + 3i g' \int \frac{d^2 l}{p^2 - m_\pi^2}$$

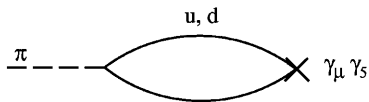


Fig. 1. Logarithmically divergent quark loop for f_π

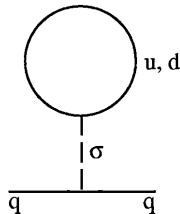


Fig. 2. Quadratically divergent graph for m_q

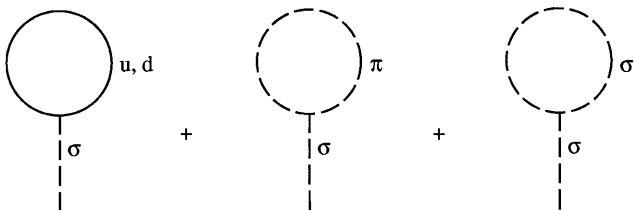


Fig. 3. Lee sum of $L\sigma M$ tadpole graphs

$$+3ig' \int \frac{d^{2l}p}{p^2 - m_\sigma^2}. \quad (4)$$

In the dimensional regularization approach these three tadpole quadratic divergences scale respectively like $m_q^2, m_\pi^2, m_\sigma^2$ in $2l = 4$ dimensions. Then using (2), one finds that (4) requires in the chiral limit [7]

$$N_c(2m_q)^4 = 3m_\sigma^4. \quad (5)$$

Moreover, the chiral anomaly (or $L\sigma M$) prediction of the $\pi^0 \rightarrow \gamma\gamma$ quark loop amplitude $F_{\pi^0\gamma\gamma} = \alpha N_c/3\pi f_\pi$, leading to a decay rate (for $N_c = 3$) of $\Gamma_{\pi^0\gamma\gamma} = m_\pi^3 |F_{\pi^0\gamma\gamma}|^2 / 64\pi \approx 7.63$ eV, quite close to the measured value of 7.74 ± 0.55 eV, empirically fixes $N_c = 3$. One then sees from (5) that the scalar meson mass

$$m_\sigma = 2m_q \quad (6)$$

has been dynamically generated (in agreement with the Nambu-Jona-Lasinio four-fermion scheme [10]). In fact in [7] the NJL relation (6) in the context of the $L\sigma M$ was obtained using a dimensional regularization lemma linking the log-divergent integral in (3) with the quadratic-divergent integral in (4) *independent of the cutoffs*.

Reversing the argument, inputting the NJL relation (6) into Lee's null tadpole condition (5) demands $N_c = 3$. This circumvents the sometimes-used large N_c limit in the discussion of possible triviality of the quark-level $L\sigma M$. Even though the three $L\sigma M$ tadpoles of Fig. 3 and (4) sum to zero, the chiral renormalization of the massless Goldstone pion is manifested by these tadpoles in a different manner. Specifically, the sum of the quark bubble and quark tadpole graphs contributing to m_π vanishes because $g' = m_\sigma^2/2f_\pi$ from (2) regardless of the implied quadratic

divergences in (4). The $L\sigma M$ version of the Goldstone theorem is then $m_\pi^2 = 0_{\text{qk loops}} + 0_{\pi \text{ loops}} + 0_{\sigma \text{ loops}} = 0$.

As for the chiral couplings, the dimensionless meson-quark coupling constant g is determined to be for $N_c = 3$ [7]

$$g = \frac{2\pi}{\sqrt{3}} \approx 3.6276, \quad (7)$$

which is compatible with the ratio $m_q/f_\pi \approx 320$ MeV/90 MeV ≈ 3.6 arising from the GTR. Alternatively, making use of the experimental couplings $g_{\pi NN} \approx 13.4$ and $g_A \approx 1.26$ [1], we may estimate $g = g_{\pi NN}/3g_A \approx 3.54$, again in a good agreement with (7). Furthermore, the study of the dynamically generated quartic meson-meson dimensionless coupling λ reveals an important link between λ and g :

$$\lambda = 2g^2 = \frac{8\pi^2}{3} \approx 26. \quad (8)$$

The relation $\lambda = 2g^2$ follows from the log-divergent gap equation (3) which “shrinks” the quark-box graph for $\pi\pi$ scattering to the quartic λ -contact interaction in the $L\sigma M$ lagrangian [7]. Alternatively, the Gell-Mann-Lévy $L\sigma M$ relation in (2) requires $\lambda = m_\sigma^2/2f_\pi^2$, which reduces to $\lambda = 2g^2$ using (6) and the GTR. Converting this λ to the dimensionless number $8\pi^2/3$ in (8) follows directly from (7).

We stress that the nonzero numerical values of the meson-quark coupling g in (7) and the meson-meson coupling λ in (8) are obtained in a manner *independent* of the implied cutoffs in (3) and (4).

It is remarkable that in the dynamically generated quark-level $L\sigma M$, the large meson-quark coupling g and larger meson-meson coupling λ are both completely driven by the fermion sector of the theory via the GTR, demanding for them fixed numerical values, (7) and (8) respectively. Also, $L\sigma M$ schemes derived from the chiral symmetry restoration temperature [11] find that $\lambda \simeq 20$ (near (8)) at zero temperature. It is important to stress that the “shrinkage” of quark loops to a point via (3) is a $Z = 0$ compositeness condition [7,12]. This $Z = 0$ condition merges the $L\sigma M$ field theory when the π and σ are treated as elementary particles with the NJL four-quark-theory when the π and σ are taken as $q\bar{q}$ bound states. In either theory $m_\pi = 0$ and $m_\sigma = 2m_q$ in the chiral limit. Moreover, meson-meson coupling g' or λ in (2) immediately leads to the σ meson decay width of ~ 700 MeV for the mass $m_\sigma = 2m_q \sim 650$ MeV.

3 Triviality of $(\lambda\phi^4)_4$ field theory

There is a strong *external* resemblance between the $L\sigma M$ and the $(\lambda\phi^4)_4$ quantum field theories. The recent studies [2-6] (also see [13]) attempting to prove the triviality of the latter theory will motivate us to investigate the question of triviality of the quark-level $L\sigma M$ field theory. First we will briefly summarize results of the $(\lambda\phi^4)_4$ references [2-6]. The lagrangian of the $(\lambda\phi^4)_4$ field theory in four

dimensional space-time is:

$$\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4, \quad (9)$$

with $\mu^2 > 0$ and $\lambda > 0$, corresponding to the spontaneously broken phase.

Also we consider the purely perturbative approach of [2-4]. Dashen and Neuberger [2] employed a perturbative (leading log) result for ultraviolet cutoff Λ

$$\frac{1}{\lambda} \gg \frac{3}{2\pi^2} \ln \frac{\Lambda}{m_\sigma}, \quad (10)$$

to obtain an upper bound on the true scalar meson mass m_σ . Lüscher and Weisz in [3] calculated the ultraviolet cutoff dependence Λ on the renormalized scalar mass and showed that the scaling laws are satisfied when

$$2m_\sigma < \Lambda < \infty. \quad (11)$$

Next, Kimura et al. [4] reproduced the Dashen-Neuberger relation (10) using the perturbative renormalization group and also by invoking nonperturbative (but approximate) Wilsonian renormalization group methods. In both cases the bound in (10) becomes a rough equality. Then combining (10) and (11), [4] deduces that $m_\sigma \leq 400$ MeV. As proposed in [2-4], such a scalar mass of order 400 MeV should be considered in an effective $(\lambda\phi^4)_4$ theory with dimensionless coupling λ (in our (8)) of order ten.

To return to the question of the triviality limit $\lambda \rightarrow 0$ as $\Lambda \rightarrow \infty$, we now consider the non-perturbative δ -parameter expansion of Bender et al. in [5,6]. Instead of a conventional perturbative treatment of the Greens functions, they propose an expansion in a power series of δ for a $\lambda(\phi^2)^{1+\delta}$ field theory in d dimensions. The latter theory, as an extension of (9), is described by the lagrangian:

$$\mathcal{L}_\delta = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}M^2\phi^2(\phi^2M^{2-d})^\delta. \quad (12)$$

Here a fixed mass parameter M has been introduced to allow the interaction term to have the correct dimensions, i.e. to keep λ *dimensionless* for arbitrary δ in any space-time dimension. Obviously, in the limit when $d \rightarrow 4$ and $\delta \rightarrow 1$, (12) reduces to (9) and the parameter M cancels out. Then one can show [5] that the n -point Green's functions can be expanded as a perturbation series in powers of δ :

$$G^{(n)}(x_1, \dots, x_n; \delta) = \sum_{k=0}^{\infty} \delta^k g_k^{(n)}(x_1, \dots, x_n). \quad (13)$$

An advantage of this method is twofold. First, δ is the *only* parameter to be treated perturbatively and consequently the results obtained from the δ -expansion (13) are non-perturbative in the *physical* parameters of the theory (such as mass and coupling). Second, it was demonstrated [6] that when $d \geq 2$, the coefficients of $\delta^{(k)}$ in (13) are less divergent than the terms in the conventional weak-coupling expansion. However, the $g^{(n)}$ s in (13) still suffer from ultraviolet divergences and thus regularization and

renormalization of the theory based on (12) and (13) are necessary.

To regularize the divergent expressions for the physical quantities (mass and coupling), a cut-off Λ is introduced in momentum space [6]. The notion of possible "triviality" of the $(\lambda\phi^4)_4$ field theory generated by the lagrangian (12) corresponds to a renormalized coupling $\lambda_R \rightarrow 0$ as the ultraviolet cutoff $\Lambda \rightarrow \infty$, so that the theory becomes effectively free.

Bender and Jones in [6] demonstrated that the triviality of the theory (12) can only follow for $d \geq 4$ dimensions, apart from the *pathological* case when the unperturbed (scalar) mass m (defined via $m^2 = \mu^2 - \frac{1}{2}\lambda M^2$) is greater than the cutoff Λ . Stated in a reverse manner, one can infer from the Bender-Jones analysis that for a *nontrivial* $(\lambda\phi^4)_4$ theory, the cutoff Λ is bounded by the unperturbed scalar mass as

$$\Lambda < m_\sigma. \quad (14)$$

The Bender-Jones result for the triviality bound of $(\lambda\phi^4)_4$ theory, namely $m_\sigma < \Lambda$, is a more restrictive conclusion than (11) in the sense that it imposes tighter limitations on the scalar mass that could possibly generate a non-trivial theory. However, we will show in what follows that it is the non-triviality (pathological) condition (14) (rather than triviality bound $m_\sigma < \Lambda$) which in fact holds for the dynamically generated quark-level $L\sigma M$ field theory of Sect. 2.

4 Linear σ Model – a nontrivial theory

As we saw in Sect. 2, the dynamically generated quark-level $L\sigma M$ has a special feature — the scalar mass and the chiral meson couplings are entirely governed by the quark sector of the theory. Moreover, the dynamically induced $L\sigma M$ is automatically "chirally renormalized" [7] due to the $L\sigma M$ Gell-Mann-Lévy chiral couplings (2). Therefore, the concept of triviality in fact is a non-sequitur in the case of the quark-level $L\sigma M$ because the dynamically generated (renormalized) values for the chiral couplings are finite, fixed nonzero numbers much greater than unity:

$$g = \frac{2\pi}{\sqrt{3}} \quad \text{and} \quad \lambda = \frac{8\pi^2}{3}, \quad (15)$$

independent of any UV cutoff. These couplings cannot vanish under *any* circumstances provided there are fundamental fermion fields in the theory generating the GTR and these nontrivial couplings (15). Consequently, the quark-level $L\sigma M$ is not effectively free, but is instead a dynamically generated nontrivial nonperturbative field theory.

Nonetheless, one can consider splitting up the quark-level $L\sigma M$ lagrangian into "bosonic" and "fermionic" parts to study its "bosonic" piece alone (in the spirit of [2-6]) but satisfying the NJL scalar mass condition $m_\sigma = 2m_q$. The results of [2-6] that were briefly summarized in the preceding section indicate that the question of triviality

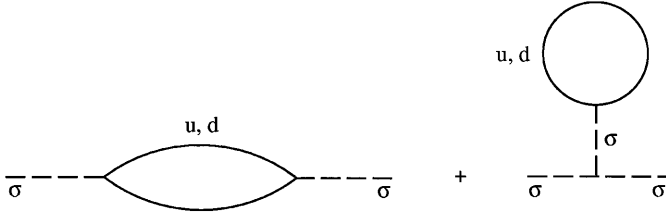


Fig. 4. Nonvanishing contributions to m_σ^2 in the quark level $L\sigma M$ to one-loop order

in a pure boson theory, such as a $(\lambda\phi^4)_4$ field theory, crucially depends on the relative scales between the ultraviolet (quadratically divergent) cutoff Λ and the scalar mass m_σ . The relations (6)-(8) for the quark level $L\sigma M$ were, however, first obtained using a dimensional regularization approach. These results are in fact independent of both ultraviolet cutoff and regularization.

To achieve consistency with a cutoff approach in the $L\sigma M$, one needs to evaluate the corresponding divergent integrals with the ultraviolet cutoff introduced. We start with the logarithmically divergent gap equation (3) due to Fig. 1. Evaluating (3) for $N_c = 3$ with a cutoff Λ yields:

$$1 = -12ig^2 \int^{\Lambda} \frac{d^4 p}{(p^2 - m_q^2)^2} = \frac{3g^2}{4\pi^2} \left[\ln \left(\frac{\Lambda^2}{m_q^2} + 1 \right) - \frac{1}{1 + (\Lambda^2/m_q^2)^{-1}} \right]. \quad (16)$$

Recalling the numerical value of the quark-meson coupling $g \approx 3.6$ (7), one sees that (16) then suggests $\Lambda^2/m_q^2 \approx 5.3$ or $\Lambda \approx 750$ MeV for $f_\pi \approx 90$ MeV and $m_q = 2\pi f_\pi/\sqrt{3} \approx 326$ MeV. This 750 MeV cutoff separates the elementary scalar mass $\sigma(650)$ from the $\bar{q}q$ bound states $\rho(770)$ and $\omega(783)$: $m_\sigma < \Lambda$ for this nontrivial $L\sigma M$ theory.

Next, we consider the quadratically divergent mass gap equation corresponding to Fig. 2 with a cutoff Λ :

$$m_q = \frac{8iN_c g^2}{m_\sigma^2} \int^{\Lambda} \frac{d^4 p m_q}{p^2 - m_q^2}, \quad (17)$$

in the spirit of the quadratically divergent cutoff approach of NJL [10]. Cancelling out the constant quark mass m_q and using the NJL or $L\sigma M$ relation $m_\sigma = 2m_q$, (17) implies:

$$1 = i \frac{24g^2}{(2m_q)^2} \int^{\Lambda} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m_q^2} = \frac{1}{2} \left[\frac{\Lambda^2}{m_q^2} - \ln \left(\frac{\Lambda^2}{m_q^2} + 1 \right) \right]. \quad (18)$$

This mass gap condition (18) leads to $\Lambda^2/m_q^2 \approx 3.5$ or $\Lambda \approx 610$ MeV for $m_q = 326$ MeV. Thus this cutoff Λ in (18) is less than $m_\sigma(652)$, whereas Λ in (16) is greater than $m_\sigma(652)$.

Although we already know in the $L\sigma M$ that $m_\sigma = 2m_q$ as obtained from either the dynamically generated theory [7] or from the Lee condition (4)-(6), we could follow NJL

(but in a $L\sigma M$ context) and simulate the scalar mass m_σ as a $\bar{q}q$ bound state by computing the quark bubble and quark tadpole Feynman graphs of Figs. 4. Such log-and quadratic-divergent graphs will be cut off in the ultraviolet region at Λ , but this will *not* be the 750 MeV cutoff of (16). Specifically for $N_f = 2$ one obtains from Figs. 4, also using $g' = m_\sigma^2/2f_\pi$ and the GTR:

$$m_\sigma^2 = 16iN_c g^2 \int^{\Lambda} d^4 p \left[\frac{1}{(p^2 - m_q^2)} - \frac{m_q^2}{(p^2 - m_q^2)^2} \right] \quad (19a)$$

$$= \frac{N_c g^2 m_q^2}{\pi^2} \frac{x^2}{1+x}, \quad (19b)$$

where $x = \Lambda^2/m_q^2$ is the (four-dimensional) dimensionless cutoff. Now invoking the meson-quark coupling (7), equations (19) reduce to

$$m_\sigma^2 = 4m_q^2 \frac{x^2}{1+x}, \quad (20a)$$

which recovers $m_\sigma = 2m_q$ if

$$x^2 = 1+x, \quad \text{or} \quad x = \frac{1+\sqrt{5}}{2} \approx 1.618. \quad (20b)$$

Then again using $m_q \approx 326$ MeV, the above cutoff of $\Lambda^2/m_q^2 \approx 1.618$ corresponds to $\Lambda \approx 415$ MeV.

Note that we have two different cutoffs. The first from the $L\sigma M$ log-divergent gap equation (3) leading to (16) and $\Lambda \approx 750$ MeV is valid when the $\sigma(650)$ meson is treated as an *elementary particle*. The second cutoff of $\Lambda \approx 610$ MeV or 415 MeV found from (18) and (20) treats the σ meson as a $\bar{q}q$ bound state. Thus it is not surprising that $\Lambda < m_\sigma$ in the latter cases; it simply means that the σ meson can no longer be treated as elementary when computed via (cutoff) quark loops from Figs. 4

Noting that the bosonic part of the quark-level $L\sigma M$ is identical with $(\lambda\phi^4)_4$ field theory provided that $\phi \equiv (\sigma, \pi)$, we can proceed further and apply the $\lambda\phi^4$ results of [5] to the quark-level $L\sigma M$. Clearly, this $L\sigma M$ field theory falls into the pathological case designated by (14): the dynamically generated scalar (σ meson, in this case) mass $m_\sigma = 2m_q \approx 652$ MeV is greater than the cutoff Λ of 415 MeV obtained from (19) and (20), or $\Lambda < m_\sigma$. Therefore, even the ‘‘bosonic’’ piece of the dynamically generated $L\sigma M$ lagrangian generates a nontrivial field theory in the sense of the Bender-Jones condition (14).

It is important to stress the relative inequality structure of (14) and not the implied absolute numerical values in (14) of $415 \text{ MeV} < 652 \text{ MeV}$. Specifically, the quark loop calculation of m_σ in (19) and (20) requires $\Lambda^2 \approx 1.618m_q^2$, while the NJL- $L\sigma M$ scalar mass squared is $m_\sigma^2 = 4m_q^2$. So the Bender-Jones pathological inequality (14) is always valid regardless of the size of m_q (i.e. $1.618 < 4$). Stated another way, the Bender-Jones triviality limit $\Lambda^2 \geq 4m_q^2$ (as opposed to (14)) can never be numerically reached from (20).

Rather than dealing with the above cutoff approach to triviality as applied to the (nontrivial) $L\sigma M$, we may instead follow [7] by starting with the chiral quark model (CQM) *massless* lagrangian

$$L_{CQM} = \bar{\psi}[i\gamma \cdot \partial + g(\sigma + i\gamma_5\tau \cdot \pi)]\psi + [(\partial\sigma)^2 + (\partial\pi)^2]/2. \quad (21)$$

Then the $L\sigma M$ lagrangian in (1) is dynamically generated by subtracting and adding quark and meson mass terms to (21). The former $-m_q$ and $-m_\sigma^2$ masses then nonperturbatively appear in the quark and meson loops of Figs. 2 and 4, while the latter $+m_q$ and $+m_\sigma^2$ arise as counterterm masses. More specifically, we change the sign of m_σ^2 on the left hand side (lhs) of (19a) and the sign of the quadratically divergent first term (17) on the rhs of (19a) as they now represent the counterterm m_σ^2, m_q masses, respectively. However the second term on the rhs of (19a) must still be computed from the log-divergent gap equation (3). Then (19a) becomes replaced by

$$-m_\sigma^2 = -2m_\sigma^2 + 4m_q^2. \quad (22)$$

The unique solution of (22) is the NJL relation $m_\sigma = 2m_q$, *independent* of any ultraviolet cutoff and any regularization scheme. To recover this NJL result in the cutoff approach of (20) requires the Bender-Jones pathological condition $\Lambda < m_\sigma$, (14).

5 Summary

Thus we must conclude that not only is the dynamically generated quark-level $L\sigma M$ quantum field theory in approximate agreement with data, but also its pure bosonic $\lambda\phi^4$ part is *nontrivial* in the sense that the coupling $\lambda \not\rightarrow 0$ as $\Lambda \rightarrow \infty$, (indeed λ is the finite number in (8)). In the language of a cutoff theory, the Bender-Jones (pathological) condition [6] $\Lambda < m_\sigma$ appears to be valid for the quark-level $L\sigma M$ when computing m_σ in the sense of [6], implying $\lambda \not\rightarrow 0$. In fact, in a dynamically generated $L\sigma M$, that is dimensionally regularized [7] with no reference to a cutoff, the bosonic coupling λ is $8\pi^2/3$, which is finite but certainly nonperturbative and nontrivial.

If instead one studied only the bosonic perturbative sector of the $L\sigma M$ with the Gell-Mann-Lévy chiral relations (2) requiring $\lambda = m_\sigma^2/2f_\pi^2$, one should not expect a small perturbative bound of unity in (1), i.e. $|\lambda/4| < 1$, to place a tight constraint on the σ mass. Rather, in the quark-level $L\sigma M$ the meson quartic coupling λ is quite large $\lambda = 8\pi^2/3 \sim 26$, and this allows m_σ to be ~ 650 MeV when $f_\pi \sim 90$ MeV (and not $m_\sigma < 400$ MeV as proposed in [2-4]). One then might expect such a large contact λ coupling to generate a correspondingly (unphysical) large $\pi\pi$ scattering length, also incompatible with Weinberg's [14] low energy PCAC analysis. Chiral symmetry, however, requires the s , t , and u channel σ poles to cancel off the dominant strength of the large $\lambda \sim 26$ contact term, thus recovering the Weinberg $\pi\pi$ scattering behavior [15].

In conclusion then, we suggest that the quark-level $L\sigma M$ driven by the GTR dynamically generates a nontrivial and large nonperturbative meson coupling $\lambda \sim 26$ which does not vanish as the cutoff $\Lambda \rightarrow \infty$. This latter field theory should be given serious consideration instead of a pure bosonic (and possibly trivial) $\lambda\phi^4$ theory, since a σ meson less than 1 GeV now appears in the particle data tables [1].

The authors are grateful for discussions with R. Delbourgo, H. F. Jones, A. Patrascioiu and for partial support from the U. S. Department of Energy.

Appendix: $L\sigma M$ Regularization Schemes

In order to convince the reader that the quark-level $L\sigma M$ is completely free of any (both logarithmic and quadratic) singularities, we review [7] for dimensional and Pauli-Villars regularization schemes. For dimensional regularization in $2l$ dimensions one expresses the log- and quadratic-divergent integrals as:

$$\int \tilde{d}^{2l}p/(p^2 - m_q^2)^2 = i\Gamma(2-l)(m_q^2)^{l-2}/(4\pi)^l, \quad (A.1)$$

$$\int \tilde{d}^{2l}p/(p^2 - m_q^2) = -i\Gamma(1-l)(m_q^2)^{l-1}/(4\pi)^l. \quad (A.2)$$

Then in the four-dimensional limit ($l \rightarrow 2$), the difference between these two divergent integrals is in fact *finite*:

$$\begin{aligned} & \int \tilde{d}^4p \left[\frac{m_q^2}{(p^2 - m_q^2)^2} - \frac{1}{p^2 - m_q^2} \right] \\ &= \lim_{l \rightarrow 2} \frac{im_q^{2l-2}}{(4\pi)^2} [\Gamma(2-l) + \Gamma(1-l)] \\ &= -\frac{im_q^2}{(4\pi)^2}, \end{aligned} \quad (A.3)$$

because of the mathematical *identity* $\Gamma(2-l) + \Gamma(1-l) \rightarrow -1$ as $l \rightarrow 2$. This dim. reg. lemma follows from the gamma function property $z\Gamma(z) = \Gamma(z+1)$.

The above dim. reg. lemma (A.3) also holds for analytic, zeta function and Pauli-Villars regularization schemes [7]. Specifically, for Pauli-Villars regularization, one expresses the difference of the log- and quadratic-divergent integrals in (A.3) as

$$\begin{aligned} & \int \tilde{d}^4p \left[\frac{m_q^2}{(p^2 - m_q^2)^2} - \frac{1}{p^2 - m_q^2} \right] \\ &= \int \frac{\tilde{d}^4p}{p^2} \left[-1 + \frac{m_q^4}{(p^2 - m_q^2)^2} \right]. \end{aligned} \quad (A.4)$$

The identity (A.4) can be verified by partial fractions of the integrands before the infinite integrals in (A.4) are performed. For Pauli-Villars regularization, introduce an ultraviolet cutoff Λ on the right-hand-side of (A.4) and

sum over auxiliary massive fermions (masses M_j) with probabilities c_j . Then (A.4) becomes:

$$\int d^4p \left[\frac{m_q^2}{(p^2 - m_q^2)^2} - \frac{1}{p^2 - m_q^2} \right] = \sum_j i c_j (\Lambda^2 - M_j^2) / (4\pi)^2. \quad (\text{A.5})$$

Applying the Pauli-Villars sum rules [16] $\sum c_j = 0$, $\sum c_j M_j^2 = m_q^2$, (A.5) reduces to $-im_q^2/(4\pi)^2$. Thus (A.5) becomes *precisely* the dim. reg. lemma (A.3), only now found from the Pauli-Villars regularization scheme.

The fact that the difference between the quadratic and the logarithmic divergences is a finite number *independent* of any particular ultraviolet cutoff Λ , leads to the cutoff-independent results $m_\sigma = 2m_q$ and $g = 2\pi/\sqrt{3}$. The latter also implies that the quartic meson coupling λ ($\lambda = 2g^2$) has a finite non-zero value which is *independent* of the ultraviolet cutoff.

References

1. C. Caso, et al., Particle Data Group, Euro. Phys. Jour. C **3**, 1 (1998); also see N. Tornqvist, M. Roos, Phys. Rev. **76**, 1575 (1996), who obtain an f_0 (400–900) scalar meson with mean mass 650 MeV
2. R. Dashen, H. Newberger, Phys. Rev. Lett. **50**, 1897 (1983)
3. M. Lüscher, D. Weisz, Nucl. Phys. B **318**, 705 (1989)
4. K. Kimura, A. Sanda, Y. Sugiyama, Mod. Phys. Lett. **A9**, 2587 (1994)
5. C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky and L. M. Simmins, Jr., Phys. Rev. D **37**, 1472 (1988); Phys. Rev. Lett. **58**, 2615 (1987)
6. C. M. Bender, H. F. Jones, Phys. Rev. D **38**, 2526 (1988)
7. R. Delbourgo, M.D. Scadron, Mod. Phys. Lett. **A10**, 251 (1995). This dimensional regularization scheme has been extended to other regularization schemes in R. Delbourgo, A.A. Rawlinson, M.D. Scadron, Mod. Phys. Lett. **A13**, 1893 (1998)
8. M. Gell-Mann, M. Lévy, Nuovo Cimento **16**, 705 (1960); also see V. De Alfaro, S. Fubini, G. Furlan, C. Rossetti, Currents in Hadron Physics Chapter 5 (North Holland, Amsterdam 1973)
9. B. W. Lee, *Chiral Dynamics* (Gordon and Breach, 1972) p. 12
10. Y. Nambu, G. Jona-Lasinio (NJL), Phys. Rev. **122**, 345 (1961)
11. K. Rajagopal, F. Wilczek, Nucl. Phys. B **404**, 577 (1993); M. Asakawa, Z. Huang, X.N. Wang, Phys. Rev. Lett. **74**, 3126 (1995), L.P. Csernai, I.N. Mishustin, *ibid*, **74**, 5005 (1995)
12. A. Salam, Nuovo Cimento **25**, 244 (1962); S. Weinberg Phys. Rev. **130**, 776 (1963). More recently see M. D. Scadron, Phys. Rev. D **57**, 5307 (1998)
13. P. Cea, et al., Lattice effective potential of massless $(\lambda\phi^4)_4$: “triviality” and spontaneous symmetry breaking, preprint BARI-TH 204/95, CS-TH 10/95, May 1995, hep-th/9505048; M. Lüscher and P. Weisz, Nucl. Phys. B **290**, 25 (1987); B. Freedman et al., Phys. Lett. **133B**, 481 (1982)
14. S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966)
15. See De Alfaro, et al., in [8] pp. 324–327
16. See e.g., C. Itzykson, J. Zuber, *Quantum Field Theory* (McGraw-Hill, 1980) p. 411